

# Instantiation of SMT problems modulo Integers (technical report)

Mnacho Echenim

Nicolas Peltier

March 2010

## Abstract

Many decision procedures for SMT problems rely more or less implicitly on an instantiation of the axioms of the theories under consideration, and differ by making use of the additional properties of each theory, in order to increase efficiency. We present a new technique for devising complete instantiation schemes on SMT problems over a combination of linear arithmetic with another theory  $\mathcal{T}$ . The method consists in first instantiating the arithmetic part of the formula, and then getting rid of the remaining variables in the problem by using an instantiation strategy which is complete for  $\mathcal{T}$ . We provide examples evidencing that not only is this technique generic (in the sense that it applies to a wide range of theories) but it is also efficient, even compared to state-of-the-art instantiation schemes for specific theories.

## 1 introduction

Research in the domain of Satisfiability Modulo Theories focuses on the design of decision procedures capable of testing the satisfiability of ground formulas modulo a given background theory. Such satisfiability checks may arise as a subproblem during the task of proving a more general formula in, e.g., software verification or interactive theorem proving. The background theories under consideration may define usual mathematical objects such as linear arithmetic, or data structures such as arrays or lists. The tools that implement these decision procedures are named SMT solvers, and they are designed to be as efficient as possible. This efficiency is obtained thanks to a sophisticated combination of state-of-the-art techniques derived from SAT solving, and ad-hoc procedures designed to handle each specific theory (see, e.g., [7] for a survey).

The lack of genericity of these theory solvers may become an issue, as additional theories, either new ones or extensions of former ones, are defined. For instance, a programmer may wish to add new axioms to the usual theory of arrays to specify, e.g., dimensions, sortedness, or definition domains. A solution to this lack of genericity was investigated in [4, 3], where a first-order theorem prover is used to solve SMT problems. Once it is proved that the theorem prover terminates on SMT problems for a given theory, it can be used as an SMT solver for that theory, and no additional implementation is required. Also, under certain conditions such as variable-inactivity (see, e.g., [3, 8]), the theorem prover can also be used as an SMT prover for a combination of theories at no further expense. However, first-order theorem provers are not capable of efficiently handling the potentially large boolean structures of SMT problems. A solution to this problem was proposed in [9], with an approach consisting of decomposing an SMT problem in such a way that the theorem prover does not need to handle its boolean part. But even with this approach, theorem provers do not seem capable to compete with state-of-the-art SMT solvers.

A new approach to handling the genericity issue consists in devising a general instantiation scheme for SMT problems. The principle of this approach is to instantiate the axioms of the theories so that it is only necessary to feed a *ground* formula to the SMT solver. The problem is then to find a way to instantiate the axioms as little as possible so that the size of the resulting formula does not blow up, and still retain completeness: the instantiated set of clauses must be satisfiable if and only if the original set is. Such an approach was investigated in [11], and an instantiation scheme was devised along with a syntactic characterization of theories for which it is refutationally complete. One theory that cannot be handled by this approach is the theory of *linear arithmetic*, which is infinitely axiomatized. Yet, this theory frequently appears in SMT problems, such as the problems on arrays with integer indices. Handling linear arithmetic is also a challenge in first-order theorem proving, and several systems have been designed to handle the

arithmetic parts of the formulas in an efficient way (see, e.g., [15] or the calculus of [2], which derives from [6]).

In this paper, we devise an instantiation scheme for theories containing particular integer constraints. This scheme, together with that of [11], permits to test the satisfiability of an SMT problem over a combination of linear arithmetic with another theory, by feeding a ground formula to an SMT solver. We show the potential efficiency of this scheme by applying it to problems in the theory of *arrays with integer indices*, and we show that it can generate sets of ground formulas that are much smaller than the ones generated by the instantiation rule of [10]. To emphasize the genericity of our approach, we also use it to integrate arithmetic constraints into a decidable subclass of many-sorted logic.

The paper is organized as follows. After recalling basic definitions from automated theorem proving, we introduce the notion of  $\mathbb{Z}$ -clauses, which are a restriction of the abstracted clauses of [6, 2], along with the inference system introduced in [2]. We define a way of instantiating integer variables in particular formulas, and show how to determine a set of terms large enough to ensure completeness of the instantiation technique on an SMT problem. We then prove that under some conditions which are fulfilled by the scheme of [11], completeness is retained after using the scheme to instantiate the remaining variables in the SMT problems. We conclude by showing how this combined scheme can be applied on concrete problems.

## 2 Preliminaries

We employ a many-sorted framework. Let  $\mathcal{S}$  denote a set of sorts, containing in particular a symbol  $\mathbb{Z}$  denoting integers. Every variable is mapped to a unique sort in  $\mathcal{S}$  and every function symbol  $f$  is mapped to a unique profile of the form  $\mathbf{s}_1 \times \dots \times \mathbf{s}_n \rightarrow \mathbf{s}$ , where  $\mathbf{s}_1, \dots, \mathbf{s}_n, \mathbf{s} \in \mathcal{S}$  (possibly with  $n = 0$ ); the sort  $\mathbf{s}$  is the *range* of the function  $f$ . Terms are built with the usual conditions of well-sortedness. The signature contains in particular the symbols  $0, -, +$  of respective profiles  $\rightarrow \mathbb{Z}, \mathbb{Z} \rightarrow \mathbb{Z}, \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ . The terms  $s^i(0)$ ,  $t + s(0)$ ,  $t + (-s(0))$  and  $t + (-s)$  are abbreviated by  $i, s(t), p(t)$  and  $t - s$  respectively. Terms (resp. variables) of sort  $\mathbb{Z}$  are called *integer terms* (resp. *integer variables*). A term is *ground* if it contains no variable. We assume that there exists at least one ground term of each sort and that for every function symbol of profile  $\mathbf{s}_1 \times \dots \times \mathbf{s}_n \rightarrow \mathbb{Z}$ , we have  $\mathbf{s}_i = \mathbb{Z}$  for all  $i \in [1..n]$ : integer terms may only have integer subterms. In other words, a noninteger term may depend on an integer term, but integer terms depend only on integer terms. This condition imposes some sort of hierarchical stratification between the theory of linear arithmetic and the other theory in which the problem is solved. As we shall see, this stratification plays a crucial role in our approach.

An *atom* is either of the form  $t \preceq s$  where  $t, s$  are two terms of sort  $\mathbb{Z}$ , or of the form  $t \simeq s$  where  $t, s$  are terms of the same sort. An atom<sup>1</sup>  $t \bowtie s$  is *arithmetic* if  $t, s$  are of sort  $\mathbb{Z}$ . A *clause* is an expression of the form  $\Gamma \rightarrow \Delta$ , where  $\Gamma, \Delta$  are sequences of non-arithmetic atoms. A *substitution*  $\sigma$  is a function mapping every variable  $x$  to a term  $x\sigma$  of the same sort. Substitution  $\sigma$  is *ground* if for every variable  $x$  in the domain of  $\sigma$ ,  $x\sigma$  is ground. For any expression  $\mathcal{E}$  (term, atom, sequence of atoms or clause),  $V(\mathcal{E})$  is the set of variables occurring in  $\mathcal{E}$  and  $\mathcal{E}\sigma$  denotes the expression obtained by replacing in  $\mathcal{E}$  every variable  $x$  in the domain of  $\sigma$  by the term  $x\sigma$ . Interpretations are defined as usual. A  $\mathbb{Z}$ -*interpretation*  $I$  is an interpretation such that the domain of sort  $\mathbb{Z}$  is the set of integers, and that the interpretation of the symbols  $0, -, +$  is defined as follows:  $I(0) = 0$ ,  $I(t + s) = I(t) + I(s)$  and  $I(-t) = -I(t)$ . A ground atom  $A$  is *satisfied* by an interpretation  $I$  if either  $A$  is of the form  $t \preceq s$  and  $I(t) \leq I(s)$  or  $A$  is of the form  $t \simeq s$  and  $I(t) = I(s)$ . A clause  $\Gamma \rightarrow \Delta$  is *satisfied* by an interpretation  $I$  if for every ground substitution  $\sigma$ , either there exists an atom  $A \in \Gamma\sigma$  that is **not** satisfied by  $I$ , or there exists an atom  $A \in \Delta\sigma$  that is satisfied by  $I$ . A set of clauses  $S$  is *satisfied* by  $I$  if  $I$  satisfies every clause in  $S$ . As usual, we write  $I \models S$  if  $S$  is satisfied by  $I$  and  $S_1 \models S_2$  if every interpretation that satisfies  $S_1$  also satisfies  $S_2$ .  $S_1$  and  $S_2$  are *equivalent* if  $S_1 \models S_2$  and  $S_2 \models S_1$ . We note  $I \models_{\mathbb{Z}} S$  if  $I$  is a  $\mathbb{Z}$ -interpretation that satisfies  $S$ ;  $S_1 \models_{\mathbb{Z}} S_2$  if every  $\mathbb{Z}$ -interpretation satisfying  $S_1$  also satisfies  $S_2$ , and  $S_1, S_2$  are  $\mathbb{Z}$ -*equivalent* if  $S_1 \models_{\mathbb{Z}} S_2$  and  $S_2 \models_{\mathbb{Z}} S_1$ .

We assume the standard notions of positions in terms, atoms and clauses. As usual, given two terms  $t$  and  $s$ ,  $t|_p$  is the subterm occurring at position  $p$  in  $t$  and  $t[s]_p$  denotes the term obtained from  $t$  by replacing the subterm at position  $p$  by  $s$ . Given an expression  $\mathcal{E}$  (term, atom, clause...), a position  $p$  is a *variable position* in  $\mathcal{E}$  if  $\mathcal{E}|_p$  is a variable.

The *flattening* operation on a set of clauses  $S$  consists in replacing non constant ground terms  $t$  occurring in  $S$  by fresh constants  $c$ , and adding to  $S$  the unit clause  $t \simeq c$ . We refer the reader to, e.g., [4] for more

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<sup>1</sup>The symbol  $\bowtie$  represents either  $\simeq$  or  $\preceq$ .

details.

### 3 $\mathbb{Z}$ -clauses

We introduce the class of  $\mathbb{Z}$ -clauses. These are restricted versions of the abstracted clauses of [6, 2], as we impose that the arithmetic constraints be represented by atoms, and not literals. We add this restriction for the sake of readability; in fact it incurs no loss of generality: for example, a literal  $\neg(a \preceq b)$  can be replaced by the  $\mathbb{Z}$ -equivalent arithmetic atom  $b \preceq p(a)$ . We present some terminology from [2], adapted to our setting.

**Definition 1** A  $\mathbb{Z}$ -clause is an expression of the form  $\Lambda \parallel \Gamma \rightarrow \Delta$ , where:

- $\Lambda$  is a sequence of arithmetic atoms (the *arithmetic part* of  $\Lambda \parallel \Gamma \rightarrow \Delta$ );
- $\Gamma \rightarrow \Delta$  is a clause such that every integer term occurring in  $\Gamma$  or in  $\Delta$  is a variable<sup>2</sup>.  $\diamond$

The property that in a  $\mathbb{Z}$ -clause  $\Lambda \parallel \Gamma \rightarrow \Delta$ , every integer term occurring in  $\Gamma$  or in  $\Delta$  is a variable is simple to ensure. If this is not the case, i.e., if  $\Gamma, \Delta$  contain an integer term  $t$  that is not a variable, then it suffices to replace every occurrence of  $t$  with a fresh integer variable  $u$ , and add the equation  $u \simeq t$  to  $\Lambda$ . This way every set of clauses can be transformed into an equivalent set of  $\mathbb{Z}$ -clauses.

The notions of position, replacement, etc. extend straightforwardly to sequences of atoms and  $\mathbb{Z}$ -clauses, taking them as terms with 3 arguments. The notion of satisfiability is extended to  $\mathbb{Z}$ -clauses as follows:

**Definition 2** A substitution  $\sigma$  is a *solution* of a sequence of arithmetic atoms  $\Lambda$  in an interpretation  $I$  if  $\sigma$  maps the variables occurring in  $\Lambda$  to integers such that  $I \models \Lambda\sigma$ . A  $\mathbb{Z}$ -clause  $\Lambda \parallel \Gamma \rightarrow \Delta$  is *satisfied* by an interpretation  $I$  if for every solution  $\sigma$  of  $\Lambda$ , the clause  $(\Gamma \rightarrow \Delta)\sigma$  is satisfied by  $I$ .  $\diamond$

Note that, although the signature may contain uninterpreted symbols of sort  $\mathbb{Z}$  (e.g. constant symbols that must be interpreted as integers), it is sufficient to instantiate the integer variables by integers only.

**Definition 3** Given a  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$ , an *abstraction atom* in  $C$  is an atom of the form  $x \simeq t$  which occurs in  $\Lambda$ .  $x \simeq t$  is *grounding* if  $t$  is ground.  $C$  is  $\mathbb{Z}$ -closed if all its integer variables occur in grounding abstraction atoms and *closed* if it is  $\mathbb{Z}$ -closed and every variable occurring in  $C$  is of sort  $\mathbb{Z}$ .  $\diamond$

Intuitively, if  $C$  is  $\mathbb{Z}$ -closed, this means that  $C$  would not contain any integer variable, had integer terms not been abstracted out. Abstraction atoms can be viewed as instantiations, as expressed by the following proposition:

**Proposition 4** Given a formula  $\phi$  such that  $V(\phi) = \{x_1, \dots, x_n\}$  and a sequence of abstraction atoms  $\Lambda = \{x_i \simeq t_i \mid i = 1, \dots, n\}$ , the sets

$$\begin{aligned} & \exists x_1 \dots \exists x_n. (\phi \wedge \Lambda) \text{ and} \\ & \phi\{x_i \leftarrow t_i \mid i = 1, \dots, n\} \end{aligned}$$

are equivalent.

**Proposition 5** Let  $C = \Lambda \parallel \Gamma \rightarrow \Delta$ , and assume  $\Lambda$  contains an abstraction atom  $x \simeq s$ . Given  $p$ , a position of  $x$  in  $\Gamma \rightarrow \Delta$ , let  $C' = C[s]_p$ , and let  $C'' = C\{x \leftarrow s\}$ . Then  $C$ ,  $C'$  and  $C''$  are equivalent.

**Example 6** Let

$$\begin{aligned} C_1 &= x \simeq a, x \simeq b \parallel \rightarrow f(x) \simeq c, g(x) \simeq c', \\ C_2 &= x \simeq a, x \simeq b \parallel \rightarrow f(x) \simeq c, g(b) \simeq c', \\ C_3 &= a \simeq a, a \simeq b \parallel \rightarrow f(a) \simeq c, g(a) \simeq c, \\ C_4 &= a \simeq a, a \simeq b \parallel \rightarrow f(a) \simeq c, g(b) \simeq c'. \end{aligned}$$

$C_2$  is obtained from  $C_1$  by replacing the occurrence of  $x$  in  $g(x)$ , i.e. the subterm at position 3.2.1.1 in  $C_1$ , by constant  $b$ ,  $C_3$  is obtained by instantiating  $C_1$  with the substitution  $\sigma = \{x \leftarrow a\}$  and  $C_4$  is obtained by instantiating  $C_2$  with  $\sigma$ . All these clauses are equivalent.  $\clubsuit$

We define an operation permitting to add arithmetic atoms to a  $\mathbb{Z}$ -clause:

**Definition 7** Consider a  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$  and a set of arithmetic atoms  $\Lambda'$ . We denote by  $[\Lambda', C]$  the  $\mathbb{Z}$ -clause  $\Lambda' \parallel \Lambda \parallel \Gamma \rightarrow \Delta$ .  $\diamond$

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<sup>2</sup>Recall that by definition a clause cannot contain arithmetic atoms.

### An inference system for $\mathbb{Z}$ -clauses.

We denote by  $\mathcal{H}$  the inference system of [2] on abstracted clauses, depicted in Figure 1. Reduction rules are also defined in [2]; the only one that is useful in our context is the *tautology deletion* rule also depicted in Figure 1. We make the additional (and natural) assumption that the ordering is such that all constants are

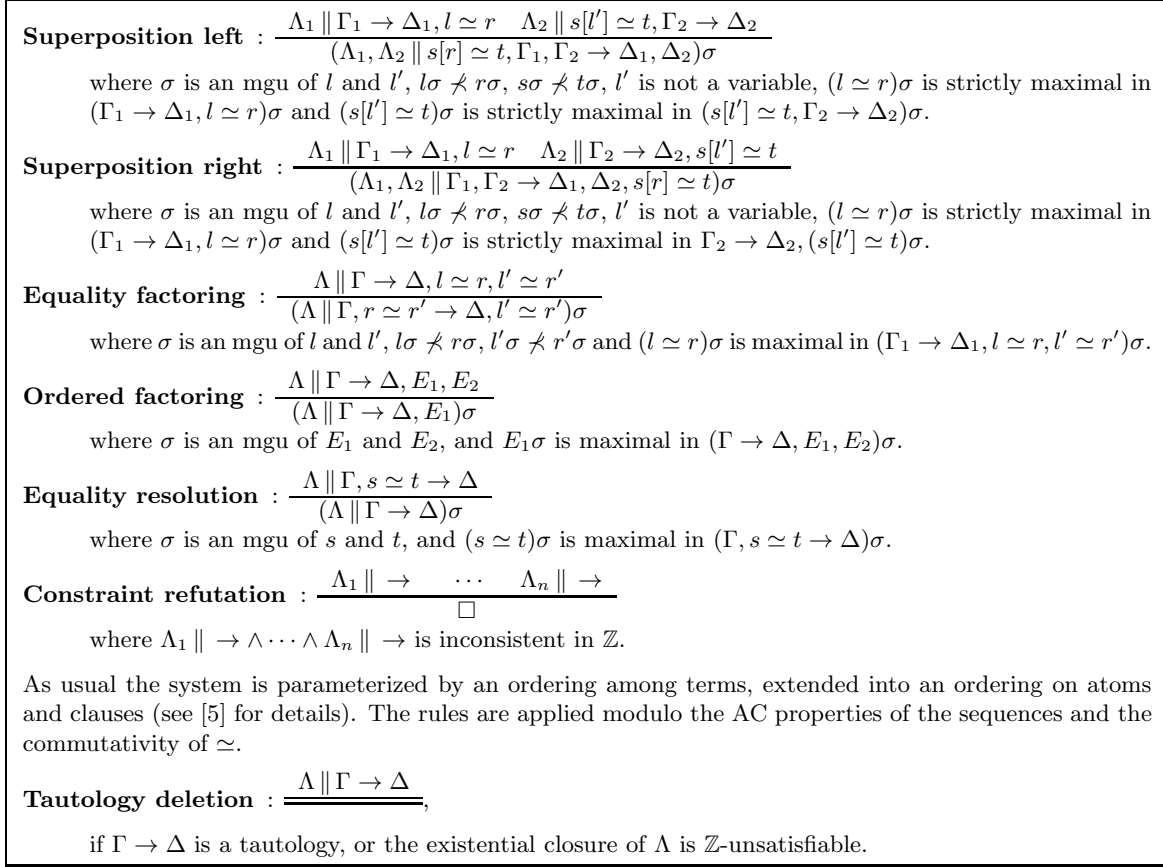


Figure 1: The inference system  $\mathcal{H}$

smaller than all non-flat terms. In order to obtain a refutational completeness result on this calculus, the authors of [6, 2] impose the condition of *sufficient completeness* on sets of clauses. Without this condition, we have the following result, stating a weaker version of refutational completeness for the calculus.

**Theorem 8** *Let  $S$  denote a  $\mathbb{Z}$ -unsatisfiable set of  $\mathbb{Z}$ -clauses. Then there exists a  $\mathbb{Z}$ -unsatisfiable set of clauses  $\{\Lambda_i \parallel \rightarrow \mid i \in \mathbb{N}\}$  such that for every  $i \in \mathbb{N}$ ,  $\Lambda_i \parallel \rightarrow$  can be deduced from  $S$  by applying the rules in  $\mathcal{H}$ .*

PROOF. (Sketch) Let  $I$  be an interpretation of the integer symbols in  $S$ , and let  $(a_i)_{i \in \mathbb{Z}}$  be a family of constant symbols of a new sort  $\mathbb{Z}'$ . We denote by  $S'$  the set of clauses of the form  $C\sigma \downarrow_I$  where:

- $\Lambda \parallel C \in S$ .
- $\sigma$  is a substitution mapping every integer variable in  $C$  to a ground integer term  $s^k(0)$  or  $-s^k(0)$  such that  $I \models \Lambda\sigma$ .
- $C\sigma \downarrow_I$  is obtained from  $C\sigma$  by replacing every ground integer term  $t$  by  $a_{I(t)}$ .

It is clear that  $S'$  is a set of clauses ( $\mathbb{Z}$  is replaced by  $\mathbb{Z}'$  in the profile of the function symbols), that contains no integer term. Furthermore,  $S'$  is unsatisfiable: if  $S'$  admits a model  $J$ , then a model  $K$  of  $S$  can be constructed by extending the interpretation  $I$  as follows: for every function  $f$  of profile  $\mathbf{s}_1 \times \dots \times \mathbf{s}_n \rightarrow \mathbf{s}$  where  $\mathbf{s} \neq \mathbb{Z}$ ,  $f^K(d_1, \dots, d_n) = f^J(d'_1, \dots, d'_n)$  where  $d'_i = d_i$  if  $\mathbf{s}_i \neq \mathbb{Z}$  and  $d'_i = J(a_{d_i})$  if  $\mathbf{s}_i = \mathbb{Z}$ . It is straightforward to check that  $K \models S$  if  $J \models S'$ .

By the refutational completeness of the superposition calculus (on clauses not containing integer terms),  $S'$  admits a refutation. Note that by definition, no superposition within terms of sort  $\mathbb{Z}'$  can occur in  $S'$ . Thus by the usual lifting argument, this derivation can be transformed into a derivation from  $S$ : it suffices to replace in the clauses  $C\sigma$  each constant  $a_i$  by the corresponding integer variable in  $C$  and to attach the original arithmetic constraint  $\Lambda$  to the clause. This derivation yields a clause of the form  $\Lambda \parallel \rightarrow$  where  $I \models \exists \vec{x}. \Lambda$  (if  $\vec{x}$  denotes the vector of variables in  $\Lambda$ ). By repeating this for every possible interpretation, we obtain a set of clauses satisfying the desired property. The conjunction of these clauses is unsatisfiable since every arithmetic interpretation falsifies at least one clause. ■

Note that this does *not* imply refutational completeness, since the set  $\{\Lambda_i \parallel \rightarrow \mid i \in \mathbb{N}\}$  may be infinite (if this set is finite then the **Constraint refutation** rule applies and generates  $\square$ ). For instance, the set of  $\mathbb{Z}$ -clauses  $S = \{x \simeq a \parallel p(x) \rightarrow, x \simeq s(y) \parallel p(x) \rightarrow p(y), p(0), a < 0 \parallel \rightarrow\}$  is clearly unsatisfiable, and the calculus generates an infinite number of clauses of the form  $s^k(0) \simeq a \parallel \rightarrow$ , for  $k \in \mathbb{N}$ . It is actually simple to see that there is no refutationally complete calculus for sets of  $\mathbb{Z}$ -clauses, since we explicitly assume that  $\mathbb{Z}$  is interpreted as the set of integers. In our case however there are additional conditions on the arithmetic constraints that ensure that only a finite set of  $\mathbb{Z}$ -clauses of the form  $\Lambda \parallel \rightarrow$  will be generated. Thus, for the  $\mathbb{Z}$ -clauses we consider, refutational completeness of the calculus will hold, and it will always generate the empty clause starting from an unsatisfiable set of  $\mathbb{Z}$ -clauses. However, we do not intend to use this result to test the satisfiability of the formulas. The reason is that – as explained in the introduction – the superposition calculus is not well adapted to handle efficiently very large propositional formulas. In this paper, we use the inference system  $\mathcal{H}$  only as a theoretical tool to show the *existence* of an instantiation scheme. To this aim we need the following property (see [6], Lemma 8 for details):

**Proposition 9** *If  $\sigma$  is an mgu occurring in a  $\mathcal{H}$ -inference, then  $\sigma$  maps integer variables to integer variables.*

## 4 Instantiation of inequality formulas

Given an SMT problem over a combination of a given theory with the theory of linear arithmetic, the inference system of [2] permits to separate the reasoning on the theory itself from the reasoning on the arithmetic part of the formula. If the input set of clauses is unsatisfiable, then the inference system will generate a set of clauses of the form  $\{\Lambda_1 \parallel \rightarrow, \dots, \Lambda_n \parallel \rightarrow, \dots\}$ , which is inconsistent in  $\mathbb{Z}$ . In this section, we investigate how to safely instantiate the  $\Lambda_i$ 's, under some condition on the atoms they contain. We shall impose that each  $\Lambda_i$  be equivalent to a formula of the following form:

**Definition 10** An *inequality formula* is of the form  $\phi : \bigwedge_{i=1}^m s_i \preceq t_i$ , where for all  $i = 1, \dots, m$ ,  $s_i$  and  $t_i$  are ground terms or variables. ◇

If  $A$  is a set of terms, we use the notation  $A \preceq x$  (resp.  $x \preceq A$ ) as a shorthand for  $\bigwedge_{s \in A} s \preceq x$  (resp.  $\bigwedge_{s \in A} x \preceq s$ ). We denote by  $U_x^\phi$  the set  $U_x^\phi = \{y \in V(\phi) \mid x \preceq y \text{ occurs in } \phi\}$ . We may thus rewrite the formula  $\phi$  as

$$\phi : \bigwedge_{x \in V(\phi)} (A_x^\phi \preceq x \wedge x \preceq B_x^\phi \wedge \bigwedge_{y \in U_x^\phi} x \preceq y) \wedge \psi,$$

where the sets  $A_x^\phi$  and  $B_x^\phi$  are ground for all  $x$ , and  $\psi$  only contains inequalities between ground terms.

**Definition 11** For all  $x \in V(\phi)$ , we consider the sets  $\bar{B}_x^\phi$ , defined as the smallest sets satisfying the following property:

$$\bar{B}_x^\phi \supseteq B_x^\phi \cup \bigcup_{y \in U_x^\phi} \bar{B}_y^\phi \cup \{\chi\},$$

where  $\chi$  is a special constant that does not occur in  $\phi$ . ◇

Note that the  $\bar{B}_x^\phi$ 's are never empty.

**Example 12** Consider the formula  $\phi$ :

$$\begin{aligned} a_1 \preceq x \quad \wedge \quad x \preceq b_1 \quad \wedge \quad x \preceq y \quad \wedge \quad x \preceq z \\ a_2 \preceq y \quad \wedge \quad y \preceq b_2 \quad \wedge \quad y \preceq b_3 \quad \wedge \quad a_3 \preceq z. \end{aligned}$$

Then  $\bar{B}_x^\phi = \{b_1, b_2, b_3, \chi\}$ ,  $\bar{B}_y^\phi = \{b_2, b_3, \chi\}$ , and  $\bar{B}_z^\phi = \{\chi\}$ . ♣

**Proposition 13** For every inequality formula  $\phi$ , every variable  $x$  occurring in  $\phi$  and for every term  $t \in \bar{B}_x^\phi$ , either  $t = \chi$  or  $\phi \models_{\mathbb{Z}} x \preceq t$ .

PROOF. Let  $C_x$  be the set of terms  $t$  such that either  $t = \chi$  or  $\phi \models_{\mathbb{Z}} x \preceq t$ . By definition,  $C_x$  contains  $B_x^\phi$  and  $\chi$ . Furthermore, if  $y \in U_x^\phi$  then  $x \preceq y$  occurs in  $\phi$  thus  $\phi \models_{\mathbb{Z}} x \preceq y$  hence  $\phi \models_{\mathbb{Z}} x \preceq t$ , for every  $t \in C_y$  distinct from  $\chi$  (by transitivity of  $\preceq$  in  $\mathbb{Z}$ ). Thus  $C_x \supseteq \bigcup_{y \in U_x^\phi} C_y$  and finally  $C_x \supseteq B_x^\phi \cup \bigcup_{y \in U_x^\phi} C_y \cup \{\chi\}$ . By definition the  $\bar{B}_x^\phi$ 's are the smallest sets satisfying the previous property, thus  $\forall x \in V(\phi), C_x^\phi \supseteq \bar{B}_x^\phi$  and the proof is completed. ■

**Theorem 14** Given an inequality formula  $\phi$  such that  $V(\phi) = \{x_1, \dots, x_n\}$  consider the two following formulas:

$$\begin{aligned} & [\exists x_1 \dots \exists x_n. \phi] & (\alpha) \\ & \left( \bigvee_{s_1 \in \bar{B}_{x_1}^\phi} \dots \bigvee_{s_n \in \bar{B}_{x_n}^\phi} \phi\{x_i \leftarrow s_i \mid i = 1, \dots, n\} \right) & (\beta) \end{aligned}$$

Let  $I$  denote a  $\mathbb{Z}$ -interpretation of  $(\alpha)$  and  $G$  denote a ground set containing all ground terms occurring in  $\phi$ . Then  $I \models_{\mathbb{Z}} (\alpha)$  if and only if, for any extension  $J$  of  $I$  to the constant  $\chi$ ,  $J \models_{\mathbb{Z}} (\bigwedge_{t \in G} \neg(\chi \preceq t)) \Rightarrow (\beta)$ .

PROOF. First suppose all extensions of  $I$  to  $\chi$  satisfy  $(\beta)$ , and let  $J$  denote an extension of  $I$  such that for all  $t \in G$ ,  $J(\chi) > J(t)$ . Then by construction of  $(\beta)$ , there exists a substitution  $\sigma = \{x_i \leftarrow s_i \mid i = 1 \dots, n\}$ , where for all  $i = 1, \dots, n$ ,  $s_i \in \bar{B}_{x_i}^\phi$ , such that  $J \models_{\mathbb{Z}} \phi\sigma$ . It is clear that  $I \models_{\mathbb{Z}} \phi\sigma$  since  $\chi$  does not occur in  $\phi$ .

Conversely, assume that  $I \models_{\mathbb{Z}} (\alpha)$ . Let  $J$  be an extension of  $I$  such that  $\forall t \in G, J(\chi) > J(t)$ . Let  $\sigma$  be the substitution of domain  $\{x_1, \dots, x_n\}$  such that for all  $i \in [1..n]$ ,  $\sigma(x_i) = \min_{t \in \bar{B}_{x_i}^\phi} (I(t))$ . We prove that  $J \models \phi\sigma$ .

$I \models \exists x_1 \dots \exists x_n. \phi$ , thus there exists a substitution  $\theta$  mapping each variable  $x_i$  ( $1 \leq i \leq n$ ) to an integer such that  $I \models \phi\theta$ . For every atom  $t \preceq s$  occurring in  $\phi$ , we have  $I \models (t \preceq s)\theta$ . We prove that  $J \models (t \preceq s)\sigma$  by investigating the different cases:

- If  $t, s$  are two ground terms in  $\phi$ , then since  $\chi$  does not occur in  $\phi$ , we have  $I(t) = J(t)$  and  $I(s) = J(s)$  thus  $I \models (t \preceq s) \Rightarrow J \models (t \preceq s) = (t \preceq s)\sigma$ .
- If  $t$  is a variable  $x_i$  (for some  $i \in [1..n]$ ) and  $s$  is a ground term, then by definition  $s \in B_{x_i}^\phi \subseteq \bar{B}_{x_i}^\phi$ , thus by definition of  $\sigma$ , we have  $x_i\sigma \leq I(s) = J(s)$ . Hence  $J \models (t \preceq s)\sigma$ .
- If  $s$  is a variable  $x_i$  (for some  $i \in [1..n]$ ) and  $t$  is a ground term, then by Proposition 13, since  $x_i\sigma \in I(\bar{B}_{x_i}^\phi)$  we must have either  $\phi \models x_i \preceq x_i\sigma$  or  $x_i\sigma = I(\chi)$ . If  $x_i\sigma = I(\chi)$  then since  $t$  is in  $G$ , we have  $J \models_{\mathbb{Z}} (t \preceq x_i\sigma)$  by definition of  $J$ . Otherwise, by definition  $\phi \models_{\mathbb{Z}} t \preceq x_i$ , thus  $\phi \models_{\mathbb{Z}} t \preceq x_i\sigma$ , by transitivity of  $\preceq$  in  $\mathbb{Z}$ . Hence  $I \models_{\mathbb{Z}} (t \preceq x_i)\sigma$ , i.e.  $J \models_{\mathbb{Z}} (t \preceq x_i)\sigma$ .
- Finally, assume that both  $t$  and  $s$  are variables  $x_i, x_j$  respectively, where  $i, j \in [1..n]$ . Since  $x_i \preceq x_j$  occurs in  $\phi$  we have  $\bar{B}_{x_i}^\phi \supseteq \bar{B}_{x_j}^\phi$ . Thus  $\min I(\bar{B}_{x_i}^\phi) \leq \min I(\bar{B}_{x_j}^\phi)$ , hence  $x_i\sigma \leq x_j\sigma$  i.e.  $J \models (t \preceq s)\sigma$ . ■

In our case, the sets  $B_{x_i}^\phi$  will not be known, since the clauses in which  $\phi$  occurs will not be generated explicitly (see Section 5 for details). Thus we need to use an *over-approximation* of these sets:

**Definition 15** A set of ground terms  $B$  is an *upper bound* of an inequality formula  $\phi$  if for all atoms  $x \preceq t$  occurring in  $\phi$ ,  $t$  is an element of  $B$ . The set  $B$  is an *upper bound* of a set of inequality formulas if it is an upper bound of each formula. ◇

**Proposition 16** Let  $\phi$  denote an inequality formula. If  $B$  is an upper bound of  $\phi$  then for every variable  $x$  in  $\phi$ ,  $B \cup \{\chi\} \supseteq \bar{B}_x^\phi$ .

It is clear that if  $B$  is an upper bound of an inequality formula, then Theorem 14 still holds when the variables in  $\phi$  are instantiated by all the elements in  $B \uplus \{\chi\}$  instead of just those in the  $\bar{B}_x^\phi$ 's.

**Definition 17** Given an inequality formula  $\phi$  such that  $V(\phi) = \{x_1, \dots, x_m\}$  and a set of ground terms  $B$ , a *B-definition* of  $\phi$  is a set (i.e. a conjunction) of grounding abstraction atoms  $\{x_i \simeq s_i \mid i = 1, \dots, m\}$ , such that every  $s_i$  is in  $B$ . We denote by  $\Theta_B[\phi]$  the set of all  $B$ -definitions of  $\phi$ . ◇

Intuitively, a  $B$ -definition of a formula represents a grounding instantiation of this formula.

**Example 18** Let  $\phi = x \preceq f(a) \wedge b \preceq x \wedge y \preceq a$ , and assume  $B = \{a, c\}$ . Then  $\Theta_B[\phi]$  contains four sets:

- $\{x \simeq a, y \simeq a\}$ , which corresponds to substitution  $\{x \leftarrow a, y \leftarrow a\}$ ,
- $\{x \simeq a, y \simeq c\}$ , which corresponds to substitution  $\{x \leftarrow a, y \leftarrow c\}$ ,
- $\{x \simeq c, y \simeq a\}$ , which corresponds to substitution  $\{x \leftarrow c, y \leftarrow a\}$ ,
- $\{x \simeq c, y \simeq c\}$ , which corresponds to substitution  $\{x \leftarrow c, y \leftarrow c\}$ . ♣

We rephrase a direct consequence of Theorem 14 using  $B$ -definitions:

**Corollary 19** Let  $\{\phi_1, \dots, \phi_n\}$  denote a set of inequality formulas over the disjoint sets of variables  $\{x_{i,1}, \dots, x_{i,m_i} \mid i = 1, \dots, n\}$ , let  $B$  denote an upper bound of this set, and assume that  $\bigvee_{i=1}^n \exists x_{i,1} \dots \exists x_{i,m_i} \phi_i$  is valid in  $\mathbb{Z}$ . If  $G$  contains all ground terms occurring in the inequality formulas and  $B' = B \uplus \{\chi\}$ , then

$$\bigwedge_{t \in G} \neg(\chi \preceq t) \Rightarrow \bigvee_{i=1}^n \left( \bigvee_{\Lambda'_i \in \Theta_{B'}[\phi_i]} \exists x_{i,1} \dots \exists x_{i,m_i} \phi_i \wedge \Lambda'_i \right) \quad (\gamma),$$

is also valid in  $\mathbb{Z}$ .

PROOF. Let  $I$  denote a  $\mathbb{Z}$ -interpretation that interprets all the symbols in  $\Sigma \uplus \{\chi\}$ ; by hypothesis,  $I$  must satisfy the formula  $\bigvee_{i=1}^n \exists x_{i,1} \dots \exists x_{i,m_i} \phi_i$  in  $\mathbb{Z}$ . Since for all  $i = 1, \dots, n$  and for all terms  $s_{i,1}, \dots, s_{i,m_i}$ , the formulas

$$\phi_i \{x_{i,j} \leftarrow s_{i,j} \mid j = 1, \dots, m_i\} \text{ and } \exists x_{i,1} \dots \exists x_{i,m_i} \phi_i \wedge \bigwedge_{j=1}^{m_i} x_{i,j} \simeq s_{i,j}$$

are equivalent by Proposition 4, we deduce by Theorem 14 that  $I$  is also a model of  $(\gamma)$ , hence the result. ■

It is important to note that results similar to those of this section could have been proved by considering the terms occurring in atoms of the form  $t \preceq x$ , instead of those of the form  $x \preceq t$ , and considering lower bounds instead of upper bounds. This should allow to choose, depending on the problem and which sets are smaller, whether to instantiate variables using lower bounds or upper bounds.

## 5 Properties of inferences on $\mathbb{Z}$ -clauses

Corollary 19 shows how to safely get rid of integer variables in a set of inequality formulas, provided an upper bound of this set is known. The goal of this section is first to show that given an initial set of  $\mathbb{Z}$ -clauses  $S$ , such an upper bound can be determined, regardless of the inequality formulas that can be generated from  $S$ . Then we show that by instantiating the integer variables in  $S$ , it is still possible to generate all necessary instances of the inequality formulas. Thus,  $S$  and the corresponding instantiated set will be equisatisfiable.

We shall use Proposition 9 to describe several properties on  $\mathbb{Z}$ -clauses that are preserved by inferences. We first define a generalization of the notion of an upper bound of an inequality formula (see Definition 15), to the case of  $\mathbb{Z}$ -clauses.

**Definition 20** Given a  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$  and a set of ground terms  $B$ , we write  $C \preceq B$  if for all atoms  $x \preceq t \in \Lambda$ ,

- $\Lambda$  contains (not necessarily distinct) grounding abstraction atoms of the form  $x_i \simeq s_i$ ,  $i = 1, \dots, n$ ;
- there exist variable positions  $\{p_1, \dots, p_n\}$  such that variable  $x_i$  occurs at position  $p_i$ , and  $t[s_1]_{p_1} \dots [s_n]_{p_n} \in B$ . ◇

**Example 21** Let  $C = x \simeq a, y \simeq b, y \simeq c, z \preceq f(g(x, y), y) \parallel \rightarrow h(x, y, z) \simeq d$  and  $B = \{f(g(a, c), b)\}$ . Then  $C \preceq B$ . ♣

Intuitively, for a  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$ , the set  $B$  is an upper bound of the inequality atoms in  $\Lambda$  provided for all atoms  $x \preceq t$ , the variables in  $t$  are replaced by the correct terms. This property is preserved by inferences:

**Lemma 22** Let  $D, D'$  denote (not necessarily distinct)  $\mathbb{Z}$ -clauses, that generate a  $\mathbb{Z}$ -clause  $C$ , and let  $B$  denote a set of ground terms. If  $D \sqsubseteq B$  and  $D' \sqsubseteq B$ , then  $C \sqsubseteq B$ .

PROOF. Let  $D = \Lambda_1 \parallel \Gamma \rightarrow \Delta_1$  and  $D' = \Lambda_2 \parallel \Gamma \rightarrow \Delta_2$ . Then  $C$  is of the form  $(\Lambda_1, \Lambda_2 \parallel \Gamma \rightarrow \Delta)\sigma$ , where  $\sigma$  maps integer variables to integer variables by Proposition 9. Let  $x \preceq t$  denote an atom in  $(\Lambda_1, \Lambda_2)\sigma$ , then one of  $\Lambda_1, \Lambda_2$ , say  $\Lambda_1$ , contains an atom  $y \preceq t'$ , such that  $(y \preceq t')\sigma = x \preceq t$ . Since  $D \sqsubseteq B$ , by hypothesis,  $\Lambda_1$  contains grounding abstraction atoms of the form  $x_i \simeq s_i$  and there exists variable positions  $\{p_1, \dots, p_n\}$  such that  $t'[s_1]_{p_1} \dots [s_n]_{p_n} \in B$ . It is clear that for all  $i$ ,  $(x_i \simeq s_i)\sigma$  is a grounding abstraction atom in  $\Lambda_1\sigma$ , thus  $t[s_1]_{p_1} \dots [s_n]_{p_n} = t'[s_1]_{p_1} \dots [s_n]_{p_n} \in B$ . ■

In order not to unnecessarily instantiate some of the integer variables in a  $\mathbb{Z}$ -clause, we distinguish those that appear in abstraction atoms from those that appear in inequality atoms. It will only be necessary to instantiate the latter variables.

**Definition 23** Let  $C = \Lambda \parallel \Gamma \rightarrow \Delta$ ; the set of *abstraction variables in  $C$*   $V_{\text{abs}}(C)$  and the set of *inequality variables in  $C$*   $V_{\text{ineq}}(C)$  are defined as follows:

$$V_{\text{abs}}(C) = \{x \in V(C) \mid \Lambda \text{ contains an abstraction atom } x \simeq t\}$$

and

$$V_{\text{ineq}}(C) = \{x \in V(C) \mid \Lambda \text{ contains a atom of the form } x \preceq t \text{ or } t \preceq x\}.$$

We may assume without loss of generality that all integer variables in a  $\mathbb{Z}$ -clause  $C$  are in  $V_{\text{abs}}(C) \cup V_{\text{ineq}}(C)$ . If this is not the case, it suffices to add to the arithmetic part of  $C$  the atom  $x \preceq x$ .

We define the notion of a *preconstrained  $\mathbb{Z}$ -clause*. If a preconstrained  $\mathbb{Z}$ -clause is of the form  $\Lambda \parallel \rightarrow$ , then  $\Lambda$  will be equivalent to an inequality formula, and this property is preserved by inferences.

**Definition 24** A  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$  is *preconstrained* if every atom in  $\Lambda$  that is not a grounding abstraction atom either has all its variables in  $V_{\text{abs}}(C)$ , or is of the form  $x \preceq t$  or  $t \preceq x$ , where  $t$  is either a variable itself, or has all its variables in  $V_{\text{abs}}(C)$ . ◇

**Example 25**  $x \simeq a, y \simeq b, f(x, y) \simeq g(y), z \preceq g(x) \parallel \rightarrow h(x, y, z) \simeq e$  is preconstrained but  $x \simeq a, y \preceq g(y) \parallel \rightarrow h(x, y, z) \simeq e$  is not because  $y$  does not occur in a grounding abstraction atom. ♣

**Lemma 26** Let  $D, D'$  denote (not necessarily distinct)  $\mathbb{Z}$ -clauses that generate a  $\mathbb{Z}$ -clause  $C$ . If  $D$  and  $D'$  are preconstrained, then so is  $C$ .

PROOF. This is a direct consequence of Proposition 9, since integer variables are mapped to integer variables by mgu  $\sigma$ , and it is simple to verify that  $V_{\text{abs}}(C) = (V_{\text{abs}}(D) \cup V_{\text{abs}}(D'))\sigma$ . ■

**Definition 27** Given a  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$ , we denote by  $T_{\text{abs}}(C)$  the set

$$\Lambda \parallel \Gamma \rightarrow \Delta = \{t \mid x \simeq t \text{ is a grounding abstraction atom in } \Lambda\}.$$

Given a set of  $\mathbb{Z}$ -clauses  $S$ , we denote by  $T_{\text{abs}}(S)$  the set  $\bigcup_{C \in S} T_{\text{abs}}(C)$ . ◇

**Lemma 28** If  $C$  is generated from a set of clauses  $S$ , then  $T_{\text{abs}}(C) \subseteq T_{\text{abs}}(S)$ .

PROOF. This is a direct consequence of Proposition 9. ■

We extend the notion of a  $B$ -definition to  $\mathbb{Z}$ -clauses. Intuitively, a  $B$ -definition of such a  $\mathbb{Z}$ -clause represents a ground instantiation of the inequality variables it contains.

**Definition 29** Given a  $\mathbb{Z}$ -clause  $C$  such that  $V_{\text{ineq}}(C) = \{x_1, \dots, x_m\}$  and a set of ground terms  $B$ , a  *$B$ -definition of  $C$*  is a set of grounding abstraction atoms  $\{x_i \simeq s_i \mid i = 1, \dots, m\}$ , such that every  $s_i$  is in  $B$ . We denote by  $\Theta_B[C]$  the set of all  $B$ -definitions of  $C$ . Given a set of  $\mathbb{Z}$ -clauses  $S$ , we denote by  $S_B$  the set  $S_B = \{[\Lambda', C] \mid C \in S \wedge \Lambda' \in \Theta_B[C]\}$ . ◇



If a  $\mathbb{Z}$ -clause  $C$  is generated from a set of  $\mathbb{Z}$ -clauses  $S$ , the following lemma shows that by carefully instantiating the inequality variables occurring in  $S$ , we obtain a set of  $\mathbb{Z}$ -clauses that generates the required instances of  $C$ .

**Lemma 30** *Consider a set of  $\mathbb{Z}$ -clauses  $S$  and a set of terms  $B$  such that  $T_{\text{abs}}(S) \subseteq B$ . If  $C = \Lambda \parallel \Gamma \rightarrow \Delta$  is generated from  $S$  and  $\Lambda' \in \Theta_B[C]$ , then  $C' = [\Lambda', C]$  can be generated from  $S_B$ .*

PROOF. By induction on the length of the derivation. Assume  $C$  is generated by a superposition from  $D_1 = \Lambda_1 \parallel \Gamma_1 \rightarrow \Delta_1$  into  $D_2 = \Lambda_2 \parallel \Gamma_2 \rightarrow \Delta_2$ ; the other cases are identical. Then  $C$  is of the form  $(\Lambda_1, \Lambda_2 \parallel \Gamma \rightarrow \Delta)\sigma$ , and  $\sigma$  maps the variables in  $\Lambda = \Lambda_1 \cup \Lambda_2$  to variables. For  $i = 1, 2$ , let

$$\Lambda'_i = \{x \simeq t \mid x \in V_{\text{ineq}}(D_i) \text{ and } x\sigma \simeq t \text{ is an abstraction atom in } (\Lambda \cup \Lambda')\sigma\},$$

Then  $C'$  is generated by  $D'_1 = [\Lambda'_1, D_1]$  and  $D'_2 = [\Lambda'_2, D_2]$ .

By Lemma 28,  $T_{\text{abs}}(C) \subseteq T_{\text{abs}}(S)$ , hence  $T_{\text{abs}}(C') \subseteq B$ . Thus, for  $i = 1, 2$ ,  $\Lambda'_i \in \Theta_B[D_i]$ ; by the induction hypothesis,  $D'_1$  and  $D'_2$  are generated by  $S_B$ , therefore, so is  $C'$ . ■

**Example 31** Consider the  $\mathbb{Z}$ -clauses  $D_1 = x \simeq i \parallel \rightarrow \text{select}(a, x) \simeq e$  and  $D_2 = y \preceq b \parallel \rightarrow \text{select}(a, y) \simeq e'$ , and let  $B = \{i\}$ . These  $\mathbb{Z}$ -clauses generate  $C = x \simeq i, x \preceq b \parallel \rightarrow e \simeq e'$ , which is also generated by  $D_1$  and  $[y \simeq i, D_2]$ . ♣

The following relation permits to keep track of the ground integer terms that may occur in a derivation:

**Definition 32** Given a  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$  and a set of ground terms  $G$ , we write  $C \sqsubseteq_{\mathbb{Z}} B$  if for all nonvariable terms  $t$  in  $\Lambda$ ,

- $\Lambda$  contains (not necessarily distinct) grounding abstraction atoms of the form  $x_i \simeq s_i$ ,  $i = 1, \dots, n$ ;
- there exist variable positions  $\{p_1, \dots, p_n\}$  such that variable  $x_i$  occurs at position  $p_i$ , and  $t[s_1]_{p_1} \dots [s_n]_{p_n} \in G$ . ◇

We prove a stability result on the set of ground integer terms that may occur in a derivation:

**Proposition 33** *Let  $D, D'$  denote (not necessarily distinct)  $\mathbb{Z}$ -clauses, that generate a  $\mathbb{Z}$ -clause  $C$ , and let  $G$  denote a set of ground terms. If  $D \sqsubseteq_{\mathbb{Z}} G$  and  $D' \sqsubseteq_{\mathbb{Z}} G$ , then  $C \sqsubseteq_{\mathbb{Z}} G$ .*

We may now state a result which links the **constraint refutation** rule of the inference system with Corollary 19, and suggests a way of safely instantiating inequality variables in a set of  $\mathbb{Z}$ -clauses.

**Lemma 34** *Let  $B$  denote a set of ground terms, and let  $S = \{C_1, \dots, C_n\}$  denote a set of  $\mathbb{Z}$ -clauses such that for all  $i = 1, \dots, n$ ,  $C_i = \Lambda_i \parallel \rightarrow$  is a pre constrained  $\mathbb{Z}$ -clause such that  $C_i \trianglelefteq B$ . Given a constant symbol  $\chi$  that does not occur in  $S$ , let  $B' = B \cup \{\chi\}$ . If  $G \supseteq B$  is a ground set such that for all,  $C_i \sqsubseteq_{\mathbb{Z}} G$ , then  $S$  is satisfiable in  $\mathbb{Z}$  if and only if*

$$\bigcup_{i=1}^n \{[\Lambda'_i, C_i] \mid \Lambda'_i \in \Theta_{B'}[C_i]\} \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$$

*is satisfiable in  $\mathbb{Z}$ .*

PROOF. For  $i = 1, \dots, n$ , let  $\{x_{i,1}, \dots, x_{i,m_i}\}$  denote the set of variables occurring in  $\Lambda_i$ . Since  $\{C_1, \dots, C_n\}$  is unsatisfiable in  $\mathbb{Z}$ , the formula  $\bigvee_{i=1}^n \exists x_{i,1} \dots \exists x_{i,m_i} \Lambda_i$  must be valid in  $\mathbb{Z}$ . Since every  $C_i$  is pre constrained and such that  $C_i \trianglelefteq B$ , every  $\Lambda_i$  is equivalent to an inequality formula of the form

$$\phi_i \equiv \bigwedge x_j \preceq s'_j \wedge \bigwedge s'_k \preceq x_k \wedge \psi',$$

over the set of variables  $\{x_{i,1}, \dots, x_{i,m_i}\}$ , and  $\phi_i$  is upper bounded by  $B$ .

By Corollary 19 the formula

$$\bigwedge_{t \in G} \neg(\chi \preceq t) \Rightarrow \bigvee_{i=1}^n \left( \bigvee_{\Lambda'_i \in \Theta_{B'}[\phi_i]} \exists x_{i,1} \dots \exists x_{i,m_i} \cdot \phi_i \wedge \Lambda'_i \right)$$

is valid in  $\mathbb{Z}$  if and only if the formula

$$\bigwedge_{t \in G} \neg(\chi \preceq t) \Rightarrow \bigvee_{i=1}^n \left( \bigvee_{\Lambda'_i \in \Theta_{B'}[C_i]} (\exists x_{i,1} \cdots \exists x_{i,m_i} \cdot \Lambda_i \wedge \Lambda'_i) \right)$$

is valid in  $\mathbb{Z}$ , hence the result.  $\blacksquare$

We therefore obtain the main result of this section:

**Theorem 35** *Suppose  $S$  is a set of  $\mathbb{Z}$ -clauses and  $B$  is a set of ground terms such that for every  $\mathbb{Z}$ -clause of the form  $C = \Lambda \parallel \rightarrow$  generated from  $S$ :*

- $C$  is prestrained,
- $C \preceq B$ .

*Let  $B' = B \cup \{\chi\}$ , where  $\chi$  does not occur in  $S$ , and let  $G \supseteq B$  denote a set of ground terms such that for all  $C \in S$ ,  $C \subseteq_{\mathbb{Z}} G$ . Then  $S$  is  $\mathbb{Z}$ -satisfiable if and only if  $S_{B'} \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$  is  $\mathbb{Z}$ -satisfiable.*

PROOF. If  $S$  is unsatisfiable, then by Theorem 8, it generates an unsatisfiable set of  $\mathbb{Z}$ -clauses  $\{C_i \mid i \in \mathbb{N}\}$ , and each  $C_i$  is of the form  $\Lambda_i \parallel \rightarrow$ . The number of such  $\mathbb{Z}$ -clauses  $S$  can generate is finite, since only a finite number of terms can appear in the arithmetic part of each  $\mathbb{Z}$ -clause, hence the **constraint refutation** rule can be applied to generate the empty  $\mathbb{Z}$ -clause. By Lemma 34,  $S_{B'} \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$  is unsatisfiable. Now assume  $S$  is satisfiable, then since  $\chi$  does not occur in  $S$ , it is clear that  $S \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$  is also satisfiable, hence, so is the instantiated set  $S_{B'} \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$ .  $\blacksquare$

In particular, since we may assume all the integer variables occurring in  $S$  are in a  $V_{\text{abs}}(C) \cup V_{\text{ineq}}(C)$  for some  $C \in S$ , every  $\mathbb{Z}$ -clause occurring in  $S_{B'}$  can be reduced to a  $\mathbb{Z}$ -clause that is  $\mathbb{Z}$ -closed, and  $S_{B'} \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$  can be reduced to a set of clauses containing no integer variable. Hence, Theorem 35 provides a way of getting rid of all integer variables in a formula.

The instantiated set  $S_{B'} \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$  can further be reduced: since  $\chi$  is strictly greater than any ground term  $t$  occurring in  $S$  or in  $B$ , every atom of the form  $\chi \preceq t$  or  $t \preceq \chi$  can be replaced by false and true respectively. Furthermore, by construction  $\chi$  only appears at the root level in the arithmetic terms. Thus we can safely assume that  $\chi$  does not occur in the arithmetic part of the  $\mathbb{Z}$ -clause in  $S_{B'}$ . This implies that the inequations  $\chi \preceq t \parallel \rightarrow$  for  $t \in G$  are useless and can be removed. Note that the resulting set does not depend on  $G$ .

## 6 Completeness of the combined instantiation schemes

The aim of this section is to determine sufficient conditions guaranteeing that once the integer variables have been instantiated, another instantiation scheme can be applied to get rid of the remaining variables in the set of clauses under consideration.

Let  $\mathcal{C}$  denote a class of clause sets admitting an instantiation scheme, i.e., a function  $\gamma$  that maps every clause set  $S \in \mathcal{C}$  to a finite set of ground instances of clauses in  $S$ , such that  $S$  is satisfiable if and only if  $\gamma(S)$  is satisfiable. If  $\gamma(S)$  is finite, this implies that the satisfiability problem is decidable for  $\mathcal{C}$ . For every clause  $C$  in a set  $S \in \mathcal{C}$ , we denote by  $\gamma_S^C$  the set of ground substitutions  $\sigma$  such that  $C\sigma \in \gamma(S)$ . Thus, by definition,  $\gamma(S) = \{C\gamma_S^C \mid C \in S\}$ . Since  $\gamma$  is generic, we do not assume that it preserves  $\mathbb{Z}$ -satisfiability. In order to apply it in our setting, we need to make additional assumptions on the instantiation scheme under consideration.

**Definition 36** A term  $t$  is *independent* from a set of clauses  $S$  if for every non-variable term  $s$  occurring in  $S$ , if  $t$  and  $s$  are unifiable, then  $t = s$ . An instantiation scheme  $\gamma$  is *admissible* if:

1. It is monotonic, i.e.  $S \subseteq S' \Rightarrow \gamma(S) \subseteq \gamma(S')$ .
2. If  $S$  is a set of clauses and  $t, s$  are two terms independent from  $S$  then  $\gamma(S \cup \{t \simeq s\}) = \gamma(S) \cup \{t \simeq s\}$ .  $\diamond$

The first requirement is fairly intuitive, and is fulfilled by every instantiation procedure of our knowledge. The second one states that adding equalities between particular terms should not influence the instantiation scheme. This requirement is actually quite strong, as evidenced by the following example.

**Example 37** Let  $S = \{p(a, x), \neg p(b, c)\}$ . Since  $p(a, x)$  and  $p(b, c)$  are not unifiable, an instantiation scheme may not instantiate variable  $x$  at all. However, by adding the unit clause  $a \simeq b$  to this set, the instantiation scheme should instantiate  $x$  with constant  $c$ . ♣

Generic instantiation schemes such as those in [16, 17, 12] do not satisfy the second requirement. However, it is fulfilled by the one of [11].

From now on, we assume that  $\gamma$  denotes an admissible instantiation scheme. We show how to extend  $\gamma$  to sets of  $\mathbb{Z}$ -closed  $\mathbb{Z}$ -clauses. Such  $\mathbb{Z}$ -clauses are obtained as the output of the scheme devised in the previous section.

**Definition 38** A set of clauses  $S = \{\Lambda_i \parallel C_i \mid i \in [1..n]\}$  where  $\Lambda_i$  is a sequence of ground arithmetic atoms and  $C_i$  is a clause is  $\gamma$ -compatible if  $S' = \{C_1, \dots, C_n\} \in \mathcal{C}$ . In this case,  $\gamma(S)$  denotes the set of ground  $\mathbb{Z}$ -clauses  $\{\Lambda_i \parallel C_i \gamma_{S'}^{C_i} \mid i \in [1..n]\}$ . ◇

**Theorem 39** Let  $S = \{\Lambda_i \parallel C_i \mid i \in [1..n]\}$  denote a  $\gamma$ -compatible set of  $\mathbb{Z}$ -clauses, where  $\Lambda_i$  is a sequence of ground arithmetic atoms and  $C_i$  is a clause. Let  $\chi$  denote a constant not occurring in the arithmetic part of the clauses in  $S$  or in the scope of a function of range  $\mathbb{Z}$  in  $S$ , and consider a set  $G$  of ground integer terms such that  $\chi$  occurs in no term in  $G$ .

Then  $S \cup \bigcup_{t \in G} \{\chi \preceq t \parallel \rightarrow\}$  is  $\mathbb{Z}$ -satisfiable if and only if  $\gamma(S)$  is  $\mathbb{Z}$ -satisfiable.

PROOF. We denote by  $\Theta$  the set of ground integer terms in  $S$  that do not contain  $\chi$ , and by  $T_{\mathbb{Z}}(S)$  the set of integer terms  $t$  such that  $t$  occurs in  $S$  as an argument of a function whose range is distinct from  $\mathbb{Z}$ . By construction,  $T_{\mathbb{Z}}(S) \subseteq \Theta$ .

If  $S$  is  $\mathbb{Z}$ -satisfiable then it is clear that  $\gamma(S)$  is  $\mathbb{Z}$ -satisfiable. Now, assume that  $\gamma(S)$  admits a  $\mathbb{Z}$ -model, which we denote by  $I$ . Let  $S'$  (resp.  $S'_\gamma$ ) be the set of clauses  $C$  such that  $S$  (resp.  $\gamma(S)$ ) contains a clause of the form  $\Lambda \parallel C$ , where  $I \models \Lambda$ . By Condition 1 on instantiation scheme  $\gamma$ ,  $\gamma(S') \subseteq \gamma(\{C_1, \dots, C_n\})$ , hence  $\gamma(S') \subseteq S'_\gamma$ . We define the set of equations

$$E = \{t \simeq s \mid t, s \in T_{\mathbb{Z}}(S), I(t) = I(s)\}.$$

Since the terms in  $T_{\mathbb{Z}}(S)$  are all ground, every term in  $T_{\mathbb{Z}}(S)$  is independent from  $S$  thus by Condition 2,  $\gamma(S') \cup E = \gamma(S' \cup E)$ . Furthermore, since  $I$  is a model of  $S'_\gamma \cup E$ , necessarily,  $\gamma(S' \cup E)$  is satisfiable; and since we assumed that the instantiation scheme  $\gamma$  is refutationally complete, so is  $S' \cup E$ . Let  $J$  denote a model of  $S' \cup E$ . This interpretation is not necessarily a  $\mathbb{Z}$ -interpretation, and we show how to construct a  $\mathbb{Z}$ -interpretation  $K$  on the same domain as  $J$  for all sorts other than  $\mathbb{Z}$ , such that  $K$  satisfies  $S$ .

Given a function  $f$  with profile  $\mathbf{s}_1 \times \dots \times \mathbf{s}_k \rightarrow \mathbf{s}$ ,  $f^K$  is defined as follows. If  $f = \chi$  then  $K(\chi)$  is an integer strictly greater than every integer  $I(t)$ , where  $t \in G \cup T_{\mathbb{Z}}(S)$ . If  $\mathbf{s} = \mathbb{Z}$  and  $f \neq \chi$  then  $f^K = f^I$ . Finally, if  $\mathbf{s} \neq \mathbb{Z}$  then for every tuple  $(d_1, \dots, d_k)$  in the domain of  $K$ ,  $f^K(d_1, \dots, d_k) = f^J(d'_1, \dots, d'_k)$  where for every  $i \in [1..k]$ :

- if  $\mathbf{s}_i \neq \mathbb{Z}$  then  $d'_i = d_i$ ;
- if  $d_i = K(\chi)$  then  $d'_i = J(\chi)$ ;
- if  $d_i = I(t)$  for some term  $t \in T_{\mathbb{Z}}(S)$  then  $d'_i = J(t)$  ( $t$  is chosen arbitrarily);
- $d'_i$  is chosen arbitrarily otherwise.

By definition of  $K$ ,  $K \models \bigwedge_{t \in G} \neg(\chi \preceq t)$ ; furthermore, the interpretations of the integer terms in  $\Theta$ , which do not contain  $\chi$ , coincide on  $K$  and  $I$ . We prove that  $K \models S$ .

Let  $i \in [1..n]$ . If  $I \not\models \Lambda_i$  then  $K \not\models \Lambda_i$  (since  $\Lambda_i$  only contains integer terms in  $\Theta$  and  $I$  and  $K$  agree on such terms), thus  $K \models \Lambda_i \parallel C_i$ . We now assume that  $I, K \models \Lambda_i$ , so that  $C_i \in S'$ . Let  $\sigma$  be a ground substitution of domain  $V(C_i)$ , we show that  $K \models C_i \sigma$ .

By definition of  $J$ ,  $J \models C_i \sigma$ , and  $C_i \sigma$  contains no arithmetic atom. Thus it suffices to prove that for every noninteger term  $t$  occurring in  $C_i \sigma$ , we have  $K(t) = J(t)$ . The proof is by induction on  $t$ . Let  $t = f(t_1, \dots, t_k)$ , by definition  $K(f(t_1, \dots, t_n)) = f^K(K(t_1), \dots, K(t_n)) = f^J(d'_1, \dots, d'_k)$  where:

- if  $t_i$  is not of sort  $\mathbb{Z}$  then  $d'_i = K(t_i)$ . By induction hypothesis,  $K(t_i) = J(t_i)$ .

- if  $t_i$  is of sort  $\mathbb{Z}$  and  $K(t_i) \neq K(\chi)$  then  $t_i$  must occur in  $T_{\mathbb{Z}}(S)$  thus  $K(t_i) = I(t_i)$ . In this case,  $d'_i = J(t)$  where  $t$  is some term in  $T_{\mathbb{Z}}(S)$  such that  $K(t_i) = I(t)$ . Since  $I \models (t \simeq t_i)$  and  $t, t_i \in T_{\mathbb{Z}}(S)$  we have  $(t \simeq t_i) \in E$  hence  $J(t) = J(t_i)$ . Thus  $K(t_i) = J(t_i)$ .
- If  $K(t_i) = K(\chi)$  then  $d'_i = J(\chi)$ . Since  $S$  contains no integer variable, every ground *integer* term in  $f(t_1, \dots, t_n)$  must already occur in  $S$ . Thus  $t_i$  must occur in  $S$  and by definition of  $K(\chi)$  we must have  $t_i = \chi$ , hence  $d'_i = J(t_i)$ .

Thus,  $f^K(K(t_1), \dots, K(t_n)) = f^J(J(t_1), \dots, J(t_n))$  and  $K(f(t_1, \dots, t_n)) = J(f(t_1, \dots, t_n))$ . This implies that  $J(t) = K(t)$  for every noninteger term  $t$  occurring in  $C_i\sigma$ . Since  $J \models C_i\sigma$ , we also have  $K \models C_i\sigma$ , which proves that  $S \cup \bigwedge_{t \in G} \neg(\chi \preceq t)$  is also satisfiable. ■

### Summary.

To summarize, starting from a set of  $\mathbb{Z}$ -clauses  $S$ :

1. The scheme devised in Section 5 is applied to instantiate all integer variables occurring in  $S$ . We obtain a  $\mathbb{Z}$ -closed set of  $\mathbb{Z}$ -clauses  $S'$ .
2.  $S'$  is processed to get rid of all clauses containing arithmetic atoms of the form  $\chi \preceq t$ , and to get rid of all atoms of the form  $t \preceq \chi$  in the remaining clauses. We obtain a set of  $\mathbb{Z}$ -clauses  $S''$ .
3. Then we apply an admissible instantiation scheme (e.g., [11])  $\gamma$  on the clausal part of the  $\mathbb{Z}$ -clauses in  $S''$  to instantiate all remaining variables. We obtain a set of closed  $\mathbb{Z}$ -clauses  $S_g$ .
4. Finally we feed an SMT-solver (that handles linear arithmetic) with  $S_g$ .

The previous results ensure that  $S$  and  $S_g$  are equisatisfiable, provided  $S''$  is compatible with  $\gamma$ . This means that the procedure can be applied to test the satisfiability of an SMT problem on the combination of linear arithmetic with, e.g., *any* of the theories that the scheme of [11] is capable of handling, which include the theories of arrays, records, or lists. Note that an efficient implementation of this scheme would not instantiate variables by  $\chi$  in clauses or literals that are afterwards deleted, but would directly apply the simplification.

Note also that simple optimizations can further be applied to reduce the size of the instantiation set. For example, given a set of clauses  $S$ , there is no need to keep in the instantiation set  $B_S$  two distinct terms  $t$  and  $s$  such that  $S \models_{\mathbb{Z}} t \simeq s$ . Thus, it is useless to store in  $B_S$  a constant  $a$  and a term  $p(s(a))$ ; if  $S$  contains a unit clause  $t \simeq a$ , there is no need for  $B_S$  to contain both  $t$  and  $a$ . Another rather obvious improvement is to use several distinct sorts interpreted as integers. Then the arithmetic variables need only to be instantiated by terms of the same sort. Our results extend straightforwardly to such settings, but we chose not to directly include these optimizations in our proofs for the sake of readability.

## 7 Applications

We now show two applications of our technique to solve satisfiability problems involving integers.

### Arrays with integer indices.

The theory of *arrays with integer indices* is axiomatized by the following set of clauses, denoted by  $\mathcal{A}_{\mathbb{Z}}$ :

$$\begin{array}{lll}
\parallel \rightarrow & \text{select}(\text{store}(x, z, v), z) & \simeq v & (a_1) \\
w \preceq p(z) \parallel \rightarrow & \text{select}(\text{store}(x, z, v), w) & \simeq \text{select}(x, w) & (a_2) \\
s(z) \preceq w \parallel \rightarrow & \text{select}(\text{store}(x, z, v), w) & \simeq \text{select}(x, w) & (a_3)
\end{array}$$

Instead of clauses  $(a_2)$  and  $(a_3)$ , the standard axiomatization of the theory of arrays contains  $w \not\preceq z \parallel \rightarrow \text{select}(\text{store}(x, z, v), w) \simeq \text{select}(x, w)$ . In order to be able to apply our scheme, we replaced atom  $w \not\preceq z$  by the disjunction  $w \preceq p(z) \vee s(z) \preceq w$ , which is equivalent in  $\mathbb{Z}$ . The standard axiomatization of the theory of arrays is saturated for the superposition calculus (see, e.g., [4]), and a similar result holds for the new axiomatization:

**Proposition 40**  $\mathcal{A}_{\mathbb{Z}}$  is saturated in  $\mathcal{H}$ .

PROOF. Any inference between the axioms generates a clause that can be deleted by the **tautology deletion** rule. ■

We consider SMT problems on arrays with integer indices of a particular kind:

**Definition 41** An  $\mathcal{A}_{\mathbb{Z}}$ -inequality problem is a set of  $\mathbb{Z}$ -clauses  $\mathcal{A}_{\mathbb{Z}} \cup S_0$  where:

- the only variables occurring in  $S_0$  are integer variables,
- all non-ground arithmetic atoms occurring in  $S_0$  that are not abstraction literals are of the form  $x \preceq t$  or  $t \preceq x$ , where  $t$  is either a variable or a ground term,
- every variable occurring in a term in  $C \in S_0$  whose head symbol is **store** must occur in a grounding abstraction literal in  $C$ . ◇

Intuitively, these conditions impose that in the corresponding set of clauses without any integer term abstracted out, the only non-ground arithmetic atoms are of the form  $x \preceq t$  or  $t \preceq x$ , and every term occurring in  $S$  whose head symbol is **store** must be ground.

**Definition 42** Consider a  $\mathbb{Z}$ -clause  $C = \Lambda \parallel \Gamma \rightarrow \Delta$ .

- $C$  is an *array property clause* if it only contains integer variables,  $\Gamma \rightarrow \Delta$  contains no occurrence of the **store** symbol, and every occurrence of the **select** symbol admits a constant as a first argument.
- $C$  is an *array write clause* if it is of the form

$$\Lambda', u \simeq i \parallel \Gamma', \text{store}(a, u, e) \simeq b \rightarrow \Delta,$$

where  $a, e, b$  and all terms in  $\Gamma, \Delta$  are flat and ground. ◇

It is simple to verify that every  $\mathcal{A}_{\mathbb{Z}}$ -inequality problem can be reduced by the flattening operation to an equisatisfiable set of clauses of the form  $\mathcal{A}_{\mathbb{Z}} \cup S_p \cup S_w$ , where:

- The clauses in  $S_p$  are array property clauses. Intuitively, the clauses in these set are used to define properties on the arrays under consideration.
- The clauses in  $S_w$  are array write clauses. Intuitively, the clauses in these set represent the write operations on the arrays under consideration.

**Proposition 43** *The following results hold:*

1. *An inference between an array property clause and an array write clause has an empty mgu, and it generates an array write clause.*
2. *There are no possible inferences between an array property clause and an axiom in  $\mathcal{A}_{\mathbb{Z}}$ .*
3. *An inference between an array write clause and an axiom in  $\mathcal{A}_{\mathbb{Z}}$  generates an array property clause.*

PROOF. 1. An array write clause is of the form

$$\Lambda, u \simeq i \parallel \Gamma, \text{store}(a, u, e) \simeq b \rightarrow \Delta,$$

where every term in  $\Gamma, \Delta$  is flat and ground. Since there can be no occurrence of **store** or of a literal  $x \simeq t$  in the array property clause, its maximal literal must be an equation between constants. Hence the mgu of the two unified terms is empty and the generated clause is an array write clause.

2. Since array property clauses do not contain any occurrence of **store** and no superposition into a variable is permitted, an inference between an array property clause and an axiom in  $\mathcal{A}_{\mathbb{Z}}$  must unify terms with **select** as a head symbol. But in this case, the unified term in the axiom must be a **select(store( $x, z, v$ ),  $w$ )**, where  $w$  may be equal to  $z$ , and since the unified term in the array property clause must be of the form **select( $a, u, t$ )** where  $a$  is a constant, these terms cannot be unifiable.

3. The only term on which the superposition rule can apply in an array write clause is of the form  $\mathbf{store}(a, u, e)$  (recall that constant symbols are strictly smaller than complex terms and that no equation on integer variables is allowed in the clauses). Since  $\mathcal{A}_{\mathbb{Z}}$  contains no occurrence of  $\mathbf{store}$  at the root level, the rule must apply from the array write clause into  $\mathcal{A}_{\mathbb{Z}}$ . Thus it replaces a term  $\mathbf{store}(x, z, v)$  occurring in the axiom by  $b$ , which implies that the first argument of  $\mathbf{select}$  is a constant. Furthermore,  $x$  and  $v$  are instantiated by constants by unification, thus the obtained clause contains no variable except for the integer variables  $z, w$ . ■

**Proposition 44** *Let  $D = \Lambda, u \simeq i \parallel \Gamma, \mathbf{store}(a, u, e) \simeq b \rightarrow \Delta$  denote an array write clause generated from  $S_p \cup S_w$ . Then  $S_w$  contains an array write clause of the form  $\Lambda', u \simeq i \parallel \Gamma', \mathbf{store}(a', u, e') \simeq b' \rightarrow \Delta'$ .*

PROOF. The result is a direct consequence of Proposition 43 (1), and is proved by induction on the length of the derivation generating  $D$ . ■

For every  $\mathcal{A}_{\mathbb{Z}}$ -inequality problem  $S$ , we define the following set of ground terms, which will be used throughout this section:

$$\begin{aligned} B_S &= \{t \text{ ground} \mid x \preceq t \text{ or } \mathbf{select}(a, t) \text{ occurs in } S\} \\ &\cup \{t' \text{ ground} \mid \mathbf{store}(a, u, e) \simeq b \text{ and } u \simeq t' \text{ occur in a same clause in } S\} \\ &\cup \{p(t') \text{ ground} \mid \mathbf{store}(a, u, e) \simeq b, u \simeq t' \text{ occur in a same clause in } S\} \end{aligned}$$

**Proposition 45** *For every clause  $C$  in  $S_p \cup S_w$ ,  $C$  is pre constrained and  $C \trianglelefteq B_S$ .*

Note that the clauses in  $\mathcal{A}_{\mathbb{Z}}$  are not pre constrained.

**Lemma 46** *Every non-redundant clause  $C$  generated from  $\mathcal{A}_{\mathbb{Z}} \cup S_p \cup S_w$  other than the clauses in  $\mathcal{A}_{\mathbb{Z}}$  is pre constrained and such that  $C \trianglelefteq B_S$ .*

PROOF. The property holds for the clauses in  $S_p \cup S_w$ . We prove the result by induction on the length of the derivation, and prove at the same time that  $C$  is either an array property clause or an array write clause; this is trivially true if  $C \in S_p \cup S_w$ . Assume  $C$  is generated by a derivation of length 1, i.e., by an inference on  $D, D'$ ; these clause are not necessarily distinct. We perform a case analysis on the properties  $D$  and  $D'$  satisfy:

**$D$  and  $D'$  are axioms in  $\mathcal{A}_{\mathbb{Z}}$ .** In this case,  $C$  is redundant by Proposition 40.

**$D$  is an array property clause and  $D' \in \mathcal{A}_{\mathbb{Z}}$ .** By Proposition 43, this case is not possible.

**$D$  is an array write clause and  $D' \in \mathcal{A}_{\mathbb{Z}}$ .** Then  $D$  is a clause of the form  $\Lambda, u \simeq i \parallel \Gamma \rightarrow \mathbf{store}(a, u, e) \simeq b, \Delta$ , where a term  $\mathbf{store}(a', i, e')$  occurs in  $S_w$  by Proposition 44. By Proposition 43 (3),  $C$  is an array property clause. Assume  $D' = (a_2)$ , the other cases are similar. Then

$$C = w \preceq p(u), u \simeq i, \Lambda \parallel \rightarrow \mathbf{select}(b, w) \simeq \mathbf{select}(a, w),$$

$C$  is therefore an array property clause such that  $C \trianglelefteq B_S$ , and it is pre constrained.

**$D$  and  $D'$  are in  $S_p \cup S_w$ .** In this case,  $C \trianglelefteq B_S$  by Lemma 22,  $C$  is pre constrained by Lemma 26, and it is either an array property clause, or an array write clause by Proposition 43. ■

By Theorem 35, if we consider the set  $B'_S$  obtained from  $B_S$  by adding a constant  $\chi$  not occurring in  $S$ , then  $S$  and  $S_{B'_S} \cup \bigcup_{t \in B_S} \{\chi \preceq t \parallel \rightarrow\}$  are equisatisfiable. We restate this result using substitutions instead of abstraction atoms:

**Lemma 47** *Let  $B'_S = B_S \cup \{\chi\}$ , let  $V$  denote the set of inequality variables occurring in clauses in  $S$ , and let  $\Omega$  denote the set of all substitutions of domain  $V$  and codomain  $B'_S$ . Then  $\mathcal{A}_{\mathbb{Z}} \cup S_0$  and  $(\mathcal{A}_{\mathbb{Z}} \cup S_0)\Omega$  are equisatisfiable.*

Since we assumed all integer variables in  $S$  are either abstraction variables or inequality variables (by otherwise adding  $x \preceq x$  to the necessary clauses), we conclude that the clauses in  $S_0\Omega$  are all ground, and the clauses in  $\mathcal{A}_{\mathbb{Z}}\Omega$  are of the form:

$$\begin{array}{lll} & \parallel & \text{select}(\text{store}(x, z, v), z) \simeq v \\ s \preceq p(z) & \parallel & \text{select}(\text{store}(x, z, v), s) \simeq \text{select}(x, s) \\ s(z) \preceq s & \parallel & \text{select}(\text{store}(x, z, v), s) \simeq \text{select}(x, s), \end{array}$$

where  $s$  is a ground term. This set of terms can be instantiated using the scheme of [11]. Thus, if  $\Omega'$  denotes the set of substitutions constructed by the instantiation scheme, by Theorem 39, the sets  $S$  and  $S\Omega\Omega'$  are equisatisfiable. The latter is ground, and its satisfiability can be tested by any SMT solver capable of handling linear arithmetic and congruence closure.

We would like to emphasize that similar theories can be handled in the same way, for instance the theory of lists, records, etc. Furthermore, other axioms can be added in the theory of arrays to express additional properties, such as sortedness (see [11] for details).

### An example.

Consider the following sets:

$$\begin{array}{ll} E &= \{l_i \preceq x_i \preceq u_i \parallel \rightarrow \text{select}(a, x_i) \simeq e_i \mid i = 1, \dots, n\}, \\ F &= \{u_i \preceq p(l_i) \parallel \rightarrow \mid i = 1, \dots, n\}, \\ G &= \{u_i \preceq p(l_{i+1}) \parallel \rightarrow \mid i = 1, \dots, n-1\}, \end{array}$$

where the  $u_i$ 's and  $l_j$ 's are constants. The  $\mathbb{Z}$ -clauses in  $E$  state that array  $a$  is constant between bounds  $l_i$  and  $u_i$ , for  $i = 1, \dots, n$ ; the  $\mathbb{Z}$ -clauses in  $F$  state that each interval has at least 1 element, and the  $\mathbb{Z}$ -clauses in  $G$  state that all the intervals have a nonempty intersection. Thus, the union of these sets entails that  $a$  is constant between bounds  $l_1$  and  $u_n$ . Let  $b$  denote the array obtained from  $a$  by writing element  $e_1$  at position  $u_{n+1}$ . If  $u_{n+1} = s(u_n)$ , then  $b$  is constant between bounds  $l_1$  and  $s(u_n)$ . Let

$$\begin{array}{ll} H &= \{x \simeq u_{n+1} \parallel \rightarrow b \simeq \text{store}(a, x, e_1), \parallel \rightarrow u' \simeq s(u_n)\} \\ &\cup \{k \preceq p(l_1) \parallel \rightarrow, u_n \preceq p(k) \parallel \rightarrow\} \\ &\cup \{\parallel \text{select}(b, k) \simeq e_1 \rightarrow\} \text{ and} \\ S_0 &= E \cup F \cup G \cup H, \end{array}$$

then  $\mathcal{A}_{\mathbb{Z}} \cup S_0$  is unsatisfiable. By applying the definition of  $B_S$  from the previous section, we obtain  $B_S = \{u_1, \dots, u_n, u_{n+1}, p(u_{n+1}), k\}$ . In the first step, all variables in  $E$  are instantiated with the elements of  $B'_S = B_S \cup \{\chi\}$ , yielding<sup>3</sup> a ground set  $E'$ . The inequality variables in the axioms of  $\mathcal{A}_{\mathbb{Z}}$  are also instantiated with the elements of  $B'_S$ , yielding a set of clauses  $A$ . Then, in the second step, the clauses in  $A$  are instantiated using the term  $\text{store}(a, u_{n+1}, e_1)$ , and we obtain a set  $A'$  containing clauses of the form

$$\begin{array}{lll} x \simeq u_{n+1} & \parallel & \text{select}(\text{store}(a, x, e_1), x) \simeq e_1, \\ x \simeq u_{n+1}, s \preceq p(x) & \parallel & \text{select}(\text{store}(a, x, e_1), s) \simeq \text{select}(a, s), \\ x \simeq u_{n+1}, s(x) \preceq s & \parallel & \text{select}(\text{store}(a, x, e_1), s) \simeq \text{select}(a, s), \end{array}$$

where  $s \in B'_S$ . Then an SMT solver is invoked on the ground set of clauses  $A' \cup E' \cup F \cup G \cup H$ . The size of this set is to be compared with the one obtained by the procedure of [10], clauses are instantiated using an index set

$$\mathcal{I} = \{l_i, u_i \mid i = 1, \dots, n\} \cup \{u_{n+1}, p(u_{n+1}), s(u_{n+1}), s(u_n), k\}.$$

There are twice as many terms in this instantiation set. It is simple to check that our procedure always generates less instances than the one of [10]. In fact, there are cases in which our method is exponentially better. The simplest example is the following: for  $i = 1, \dots, n$ , let  $A_i$  denote the atom  $\text{select}(a, x_i) \simeq c_i$ , and let  $S = \mathcal{A}_{\mathbb{Z}} \cup S_0$ , where

$$S_0 = \{a \leq x_1, \dots, a \leq x_n, b \leq y \parallel A_1, \dots, A_n \rightarrow \text{select}(t, y) \simeq b\}.$$

With this set, our instantiation scheme generates only a *unique* clause, whereas the one in [10] instantiates every  $x_i$  with  $i$  and  $j$ , yielding  $2^n$  clauses.

<sup>3</sup>in an actual implementation, the variables in  $E$  would not be instantiated with  $\chi$ .

### Stratified classes.

To show the wide range of applicability of our results, we provide another example of a domain where they can be applied. The results in this section concern decidable subclasses of first-order logic with sorts, which are investigated in [1]. We briefly review some definitions.

**Definition 48** A set of function symbols  $\Sigma$  is *stratified* if there exists a function  $\text{level}$  mapping every sort  $\mathbf{s}$  to a natural number such that for every function symbol  $f \in \Sigma$  of profile  $\mathbf{s}_1 \times \dots \times \mathbf{s}_n \rightarrow \mathbf{s}$  and for every  $i \in [1..n]$ ,  $\text{level}(\mathbf{s}) > \text{level}(\mathbf{s}_i)$ . We denote by  $T_\Sigma$  (resp.  $T_\Sigma^{\mathbf{s}}$ ) the set of ground terms built on the set of function symbols  $\Sigma$  (resp. the set of terms of sort  $\mathbf{s}$  built on  $\Sigma$ ).  $\diamond$

**Proposition 49** *Let  $\Sigma$  be a finite stratified set of function symbols. Then the set  $T_\Sigma$  is finite.*

PROOF. We show, by induction on the terms, that the depth of a term in  $T_\Sigma^{\mathbf{s}}$  is bounded by  $\text{level}(\mathbf{s})$ . This obviously implies that the number of terms in  $T_\Sigma$  is finite.

Let  $t$  be a ground term of depth  $d$ .  $t$  is of the form  $f(t_1, \dots, t_n)$  where  $f$  is a function symbol of profile  $\mathbf{s}_1 \times \dots \times \mathbf{s}_n \rightarrow \mathbf{s}$  and  $t_1, \dots, t_n$  are terms of sorts  $\mathbf{s}_1, \dots, \mathbf{s}_n$  respectively. By the induction hypothesis, the depth of  $t_1, \dots, t_n$  is bounded by  $\text{level}(\mathbf{s}_1), \dots, \text{level}(\mathbf{s}_n)$  respectively. Thus the depth of  $t$  is bounded by  $1 + \max_{i \in [1..n]}(\text{level}(\mathbf{s}_i))$ . Since the signature is stratified, we have  $\forall i \in [1..n], \text{level}(\mathbf{s}) > \text{level}(\mathbf{s}_i)$  thus  $\text{level}(\mathbf{s}) \geq 1 + \max_{i \in [1..n]}(\text{level}(\mathbf{s}_i)) \geq d$ .  $\blacksquare$

A set of clauses is in  $St_0$  if its signature is stratified. In particular, any formula in the Bernays-Schönfinkel class is in  $St_0$ . By Proposition 49,  $St_0$  admits a trivial instantiation scheme: it suffices to replace each variable by every ground term of the same sort, defined on the set of function symbols occurring in  $St_0^4$ . This instantiation scheme is obviously admissible (see Definition 36).

This instantiation scheme can be applied also to the class  $St_2$  defined in [1] as an extension of the class  $St_0$  with atoms of the form  $t \in \text{Im}[f]$ , where  $f$  is a function symbol of profile  $\mathbf{s}_1 \times \dots \times \mathbf{s}_n \rightarrow \mathbf{s}$ , meaning that  $t$  is in the image of the function  $f$ . From a semantic point of view, the atom  $t \in \text{Im}[f]$  is a shorthand for  $\exists x_1, \dots, x_n. t \simeq f(x_1, \dots, x_n)$ . To ensure decidability, for every atom of the form  $t \in \text{Im}[f]$  and for every function symbol  $g$  of the same range as  $f$ , the following properties have to be satisfied:

1.  $g$  must have the same profile as  $f$ .
2. The formula  $f(x_1, \dots, x_n) \simeq g(y_1, \dots, y_n) \Rightarrow \bigwedge_{i=1}^n x_i \simeq y_i$ , where  $n$  denotes the arity of  $f$  and  $g$ , must hold in every model of the considered formula.

In [1] it is shown that every satisfiable set in  $St_2$  admits a finite model, hence,  $St_2$  is decidable. We show that any formula in  $St_2$  can be reduced to a clause set in  $St_0$ , thus reducing satisfiability problems in  $St_2$  to satisfiability problems in  $St_0$ . We begin by showing that if  $t$  is a complex term in an atom  $t \in \text{Im}[f]$ , then under certain conditions which will be satisfied for the elements in  $St_2$ , the atom can safely be replaced by an equality atom.

**Proposition 50** *Consider an atom  $g(t_1, \dots, t_n) \in \text{Im}[f]$ , and assume that  $g$  has the same profile as  $f$ . Consider also an interpretation  $I$  such that  $I \models f(x_1, \dots, x_n) \simeq g(y_1, \dots, y_n) \Rightarrow \bigwedge_{i=1}^n x_i \simeq y_i$ . Then  $I \models g(t_1, \dots, t_n) \in \text{Im}[f]$  if and only if  $I \models g(t_1, \dots, t_n) \simeq f(t_1, \dots, t_n)$ .*

PROOF. First note that  $f(t_1, \dots, t_n)$  is a well-formed term, since  $f$  and  $g$  have the same profile by hypothesis. Furthermore, it is obvious that  $g(t_1, \dots, t_n) \simeq f(t_1, \dots, t_n) \models g(t_1, \dots, t_n) \in \text{Im}[f]$ . Now, assume that  $I \models g(t_1, \dots, t_n) \in \text{Im}[f]$ . Then there exists an extension  $I'$  of  $I$  to  $x_1, \dots, x_n$  such that  $I' \models g(t_1, \dots, t_n) \simeq f(x_1, \dots, x_n)$ . But then since  $I \models f(x_1, \dots, x_n) \simeq g(y_1, \dots, y_n) \Rightarrow \bigwedge_{i=1}^n x_i \simeq y_i$  we deduce that  $I' \models t_i \simeq x_i$ , for all  $i \in [1..n]$ . Thus  $I' \models g(t_1, \dots, t_n) \simeq f(t_1, \dots, t_n)$ , and  $I \models g(t_1, \dots, t_n) \simeq f(t_1, \dots, t_n)$  since  $I$  and  $I'$  coincide on the terms not containing  $x_1, \dots, x_n$ .  $\blacksquare$

In order to get rid of atoms of the form  $t \in \text{Im}[f]$ , we prove that in the case where  $t$  is a variable, we may assume  $t$  is interpreted as a ground term in  $T_\Sigma$ .

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<sup>4</sup>possibly enriched with some constant symbols in order to ensure that each sort is nonempty.



**Lemma 51** *Let  $S$  denote a clause set in  $St_2$  built on a stratified set of symbols  $\Sigma$  such that  $T_\Sigma^s$  is nonempty for every  $s \in \mathcal{S}$ . Let  $S$  denote a clause set in  $St_2$ . If an interpretation  $I$  is a model of  $S$ , then the restriction of  $I$  to the domains  $I(T_\Sigma^s)$  is also a model of  $S$ .*

PROOF. Let  $I$  denote a model of  $S$ , and let  $J$  be the restriction of  $I$  to the domains  $I(T_\Sigma^s)$ , for every sort  $s \in \mathcal{S}$ . It is clear that  $J$  is an interpretation and, by definition, for all  $f \in \Sigma$ ,  $f^J$  is a total function. Still by definition,  $I$  and  $J$  coincide on every term in  $T_\Sigma$ . Let  $\sigma$  denote a grounding substitution and let  $A$  denote an atom built on the symbols in  $\Sigma$ . If  $A$  is of the form  $t \simeq s$  then we have  $I(t\sigma) = J(t\sigma)$  and  $I(s\sigma) = J(s\sigma)$  by construction, thus,  $I \models (t \simeq s)\sigma$  if and only if  $J \models (t \simeq s)\sigma$ . Thus ground equational atoms have the same truth values in  $I$  and in  $J$ , and in particular, we deduce that  $J \models f(x_1, \dots, x_n) \simeq g(y_1, \dots, y_n) \Rightarrow \bigwedge_{i=1}^n x_i \simeq y_i$ , for every function symbol  $f$  occurring in a term  $\text{Im}[f]$  and for every function symbol  $g$  with the same range as  $f$ .

If  $A$  is of the form  $t \in \text{Im}[f]$ , then by definition  $t\sigma$  is of the form  $g(t_1, \dots, t_n)$  for some function symbol with the same range as  $f$ . By Proposition 50,  $t\sigma \in \text{Im}[f]$  has the same truth value in  $I$  and in  $J$  as  $t\sigma \simeq f(t_1, \dots, t_n)$ . Since  $t\sigma \simeq f(t_1, \dots, t_n)$  is a ground equational atom, it has the same truth value in  $I$  and in  $J$ . Thus  $I \models t\sigma \in \text{Im}[f]$  if and only if  $J \models t\sigma \in \text{Im}[f]$ . We deduce that for all grounding substitutions  $\sigma$ ,  $A\sigma$  has the same truth value in  $I$  and in  $J$ , and since  $I \models S$ , we conclude that  $J \models S$ . ■

Proposition 50 and Lemma 51 permit to reduce a satisfiability problem in  $St_2$  to a satisfiability problem in  $St_0$ , by getting rid of all occurrences of atoms of the form  $t \in \text{Im}[f]$ . This is done by getting rid of all occurrences of atoms of the form  $t \in \text{Im}[f]$ . One such transformation is obvious: by definition, every occurrence of the form  $t \notin \text{Im}[f]$  can be replaced by  $t \not\simeq f(x_1, \dots, x_n)$ , where the  $x_i$  are fresh variables. We now focus on the other occurrences of the atoms.

**Definition 52** Let  $S$  denote a set of clauses. We denote by  $S'$  the set of clauses obtained from  $S$  by applying the following transformation rule (using a “don’t care” nondeterministic strategy):

$$\Gamma \rightarrow \Delta, x \in \text{Im}[f] \rightsquigarrow \{x \simeq g(x_1, \dots, x_n), \Gamma \rightarrow \Delta, g(x_1, \dots, x_n) \in \text{Im}[f] \mid g \in \Sigma_f\}$$

where  $x$  is a variable,  $f$  is of arity  $n$ ,  $\Sigma_f$  denotes the set of function symbols with the same profile as  $f$  and  $x_1, \dots, x_n$  are fresh variables that are pairwise distinct. We denote by  $S \downarrow_0$  the set of clauses obtained from  $S'$  by applying the following transformation rule:  $g(x_1, \dots, x_n) \in \text{Im}[f] \rightsquigarrow g(x_1, \dots, x_n) \simeq f(x_1, \dots, x_n)$ . ◇

The first rule gets rid of atoms of the form  $x \in \text{Im}[f]$  by replacing them with atoms of the form  $t \in \text{Im}[f]$  where  $t$  is a complex term, and the second rule gets rid of these atoms. It is obvious that these rules terminate: the first one decreases the number of atoms of the form  $x \in \text{Im}[f]$  where  $x$  is a variable, and the second one decreases the number of occurrences of  $\text{Im}[f]$ . Obviously, the normal forms cannot contain atoms of the form  $t \in \text{Im}[f]$  thus they must be in  $St_0$ . The rules preserve satisfiability: Proposition 50 ensures that the second rule preserves equivalence, and Lemma 51 ensures that the first one preserves satisfiability, since it shows that we can restrict ourselves to models in which the formula

$$\forall x \exists x_1, \dots, x_n. \bigvee_{g \in \Sigma_f} x \simeq g(x_1, \dots, x_n)$$

holds. Note that Condition 1 in the definition of  $St_2$  ensures that every function symbol of the same range as  $f$  is actually in  $\Sigma_f$ . We therefore have the following result:

**Theorem 53** *Let  $S \in St_2$ , then  $S \downarrow_0 \in St_0$ . Furthermore,  $S$  is satisfiable if and only if  $S \downarrow_0$  is satisfiable.*

In particular, these results hold when one of the sorts under consideration is  $\mathbb{Z}$  and  $S$  contains  $\mathbb{Z}$ -clauses:

**Corollary 54** *If  $S \in St_2$  is a set of  $\mathbb{Z}$ -clauses, then  $S$  is  $\mathbb{Z}$ -satisfiable if and only if  $S \downarrow_0$  is  $\mathbb{Z}$ -satisfiable.*

**Theorem 55** *Consider a set of  $\mathbb{Z}$ -clauses  $S = \{\Lambda_i \parallel C_i \mid i \in [1..n]\}$  in  $St_2$  such that every  $\Lambda_i \parallel C_i$  is pre-constrained, and for every occurrence of an atom  $t \in \text{Im}[f]$ , the range of  $f$  is not of sort  $\mathbb{Z}$ . The set  $S$  is  $\mathbb{Z}$ -satisfiable if and only if  $\gamma(S \downarrow_0 \Omega)$  is  $\mathbb{Z}$ -satisfiable, where:*

- $\Omega$  is the set of substitutions of domain  $V(S)$  whose codomain is a set  $B$  such that  $\Lambda_i \parallel C_i \leq B$  for all  $i = 1, \dots, n$ ;

- $\gamma$  denotes an instantiation scheme for  $St_0$  satisfying the conditions of page 10 (e.g.  $\gamma(S) = S\Omega'$  where  $\Omega'$  is the set of substitutions of domain  $V(S)$  and of codomain  $T_\Sigma$ ).

PROOF. By Corollary 54,  $S$  and  $S\downarrow_0$  are equisatisfiable in  $\mathbb{Z}$ , and since the transformation rules of Definition 52 do not influence the arithmetic parts of the  $\mathbb{Z}$ -clauses which do not contain any atom of the form  $t \in \text{Im}[f]$ , the resulting clauses are pre constrained and upper bounded. Thus, by Theorem 35,  $S\downarrow_0$  and  $S\downarrow_0 \Omega \cup \bigcup_{t \in B} \{\chi \preceq t \parallel \rightarrow\}$  are equisatisfiable. By applying Theorem 39, we deduce that  $S\downarrow_0 \Omega \cup \bigcup_{t \in B} \{\chi \preceq t \parallel \rightarrow\}$  and  $\gamma(S\downarrow_0 \Omega)$  are equisatisfiable, hence the result. ■

Examples of specifications in the classes  $St_0$  and  $St_2$  are presented in [1]. Our results allow the integration of integer constraints into these specifications.

## 8 Discussion

In this paper we presented a way of defining an instantiation scheme for SMT problems based on a combination of linear arithmetic with another theory. The scheme consists in getting rid of the integer variables in the problem, and applying another instantiation procedure to the resulting problem. Provided the integer variables essentially occur in inequality constraints, this scheme is complete for the combination of linear arithmetic with several theories of interest to the SMT community, but also for the combination of linear arithmetic with other decidable theories such as the class  $St_2$  from [1]. The application of this scheme to the theory of arrays with integer indices shows that it can produce considerably fewer ground instances than other state-of-the-art procedures, such as that of [10]. The instantiation scheme of [11] is currently being implemented, and will be followed by a comparison with other tools on concrete examples from SMT-LIB<sup>5</sup>.

As far as further research is concerned, we intend to investigate how to generalize this procedure, in which it is imposed that functions of range  $\mathbb{Z}$  can only have integer arguments. We intend to determine how to allow other functions of range  $\mathbb{Z}$  while preserving completeness. It is shown in [10] that considering arrays with integer elements, for which nested reads can be allowed, gives rise to undecidable problems, but we expect to define decidable subclasses, that may generalize those in [13]. Dealing with more general functions of range  $\mathbb{Z}$  should also allow us to devise a new decision procedure for the class of arrays with dimension that is considered in [14]. We also intend to generalize our approach to other combinations of theories that do not necessarily involve linear arithmetic, by determining conditions that guarantee *combinations* of instantiation schemes can safely be employed to eliminate all variables from a formula. Another interesting line of research would be to avoid a systematic grounding of integer variables and to use decision procedures for non-ground systems of arithmetic formulae. The main difficulty is of course that with our current approach, instantiating integer variables is required to determine how to instantiate the remaining variables.

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<sup>5</sup><http://www.smt-lib.org/>

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